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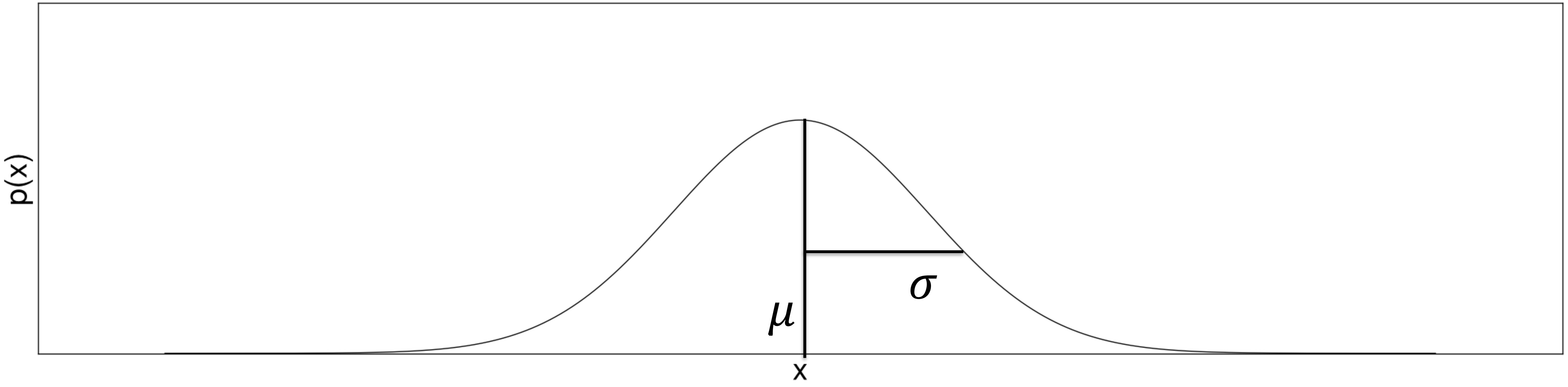
Multivariate normal distribution

The univariate normal distribution

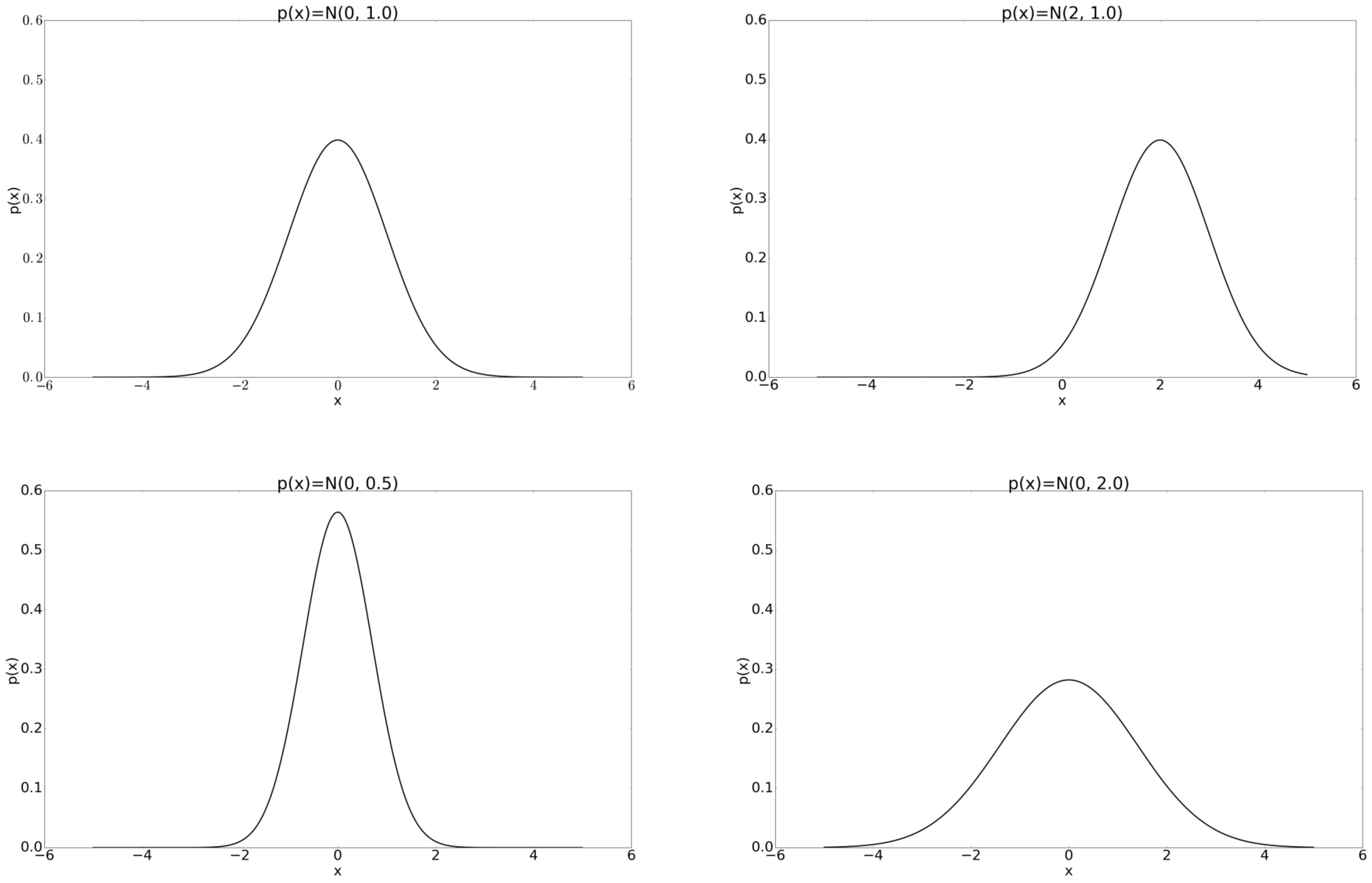
Let x be a normally distributed random variable.

$$x \sim N(\mu, \sigma^2) \text{ or } p(x) = N(\mu, \sigma^2)$$

- Mean μ: Location of the distribution
- Variance σ^2: how spread out the values are around the mean



The univariate normal distribution



Properties:

- Distribution is unimodal
- Symmetric and centered around mean
- Values far from the mean quickly become unlikely

The multivariate normal distribution

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Let $x_1, \ldots, x_n$ be jointly normally distributed random variables.

$$\mathrm{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim N(\mu, \Sigma)$$

- Mean vector $\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$: Location of the distribution

- Covariance matrix $\Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{pmatrix}$: Shape of the distribution

The covariance matrix Σ needs to be symmetric and positive definite.

Density function

Notation:

$\mathrm{p}(\mathrm{x}) = N(\mu, \Sigma)$ or $p(x_1, \dots, x_n) = N(\mu, \Sigma)$

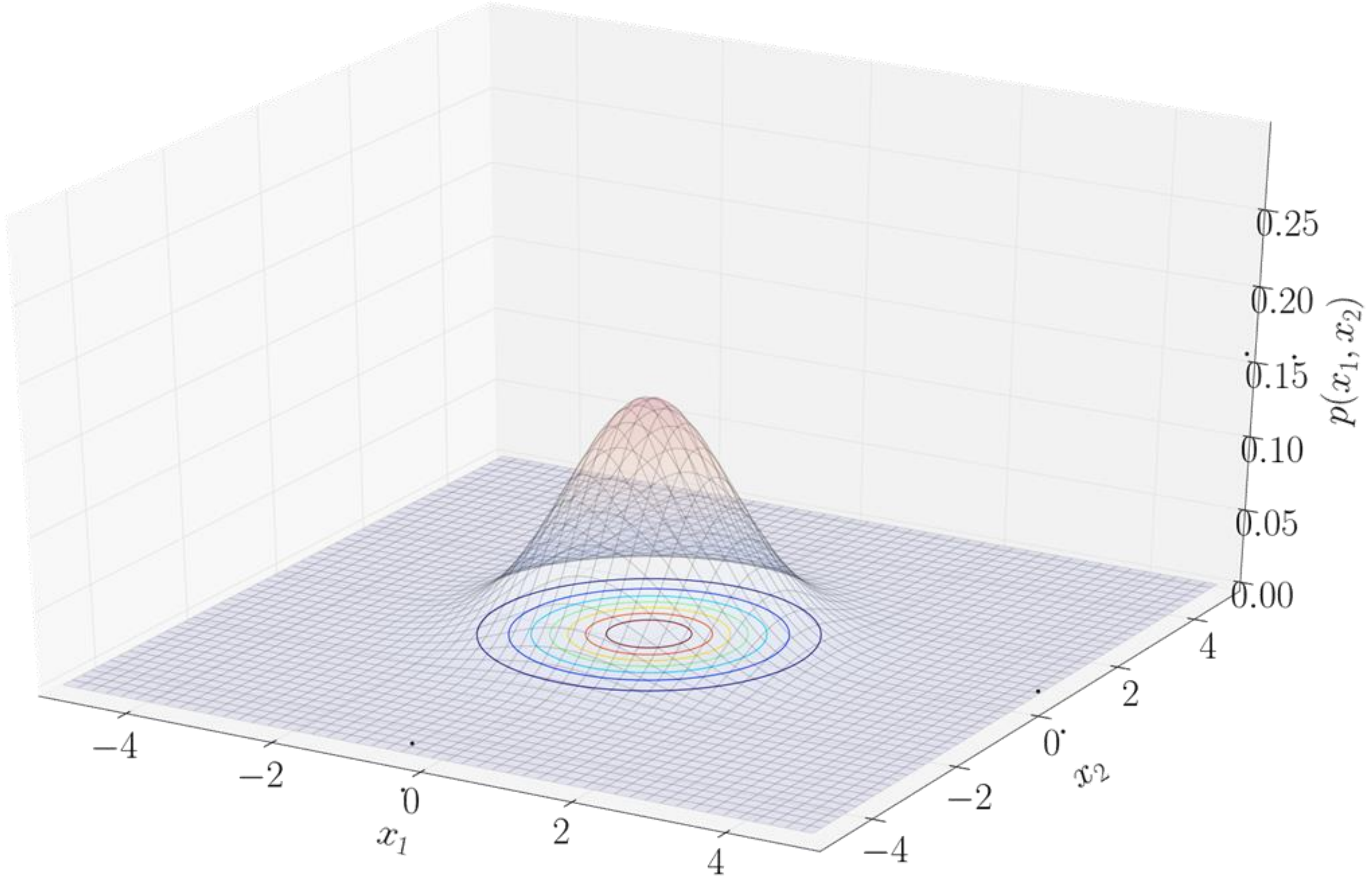
$$p(x) = \underbrace{\frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}}}_{\text{Normalization}} \exp\left(-\frac{1}{2}\underbrace{[(x - \mu)^T \Sigma^{-1} (x - \mu)]}_{\text{Mahalanobis distance}}\right)$$

Example: Bivariate normal

Let x_1, x_2 be jointly normally distributed random variables.

The bivariate normal is defined by

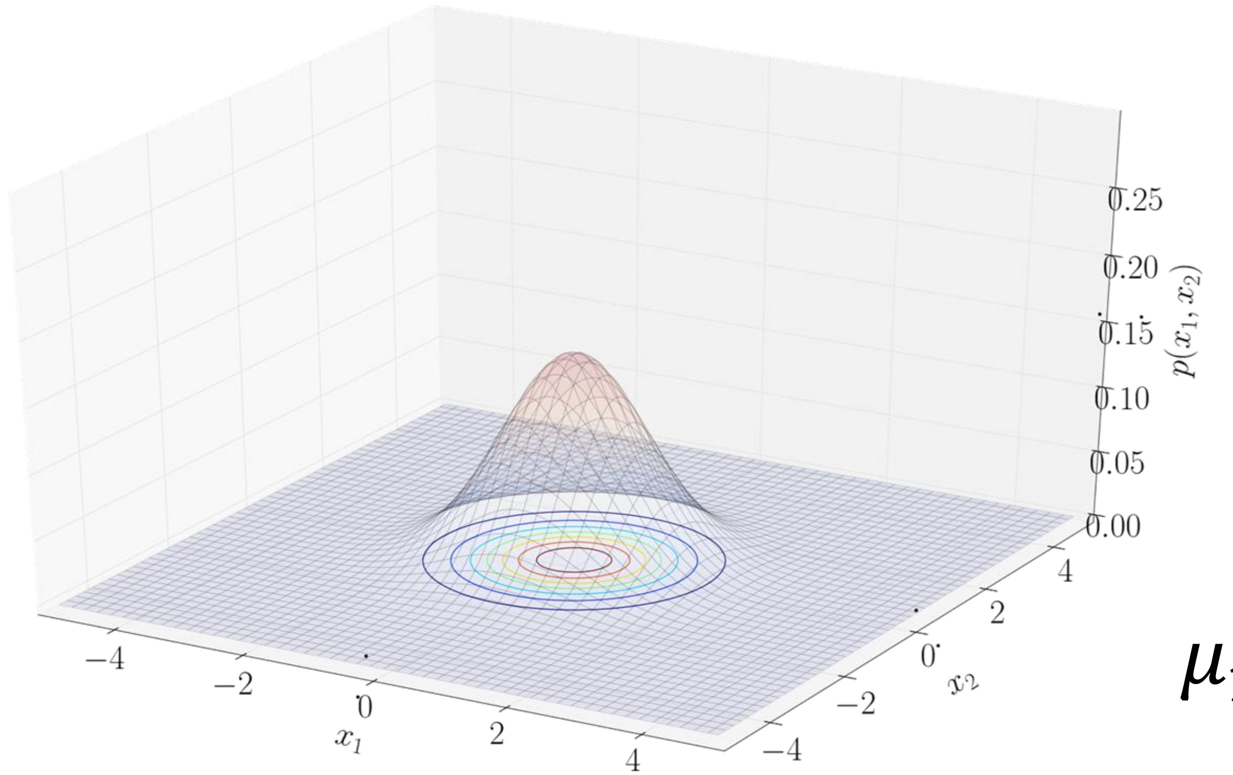
$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$



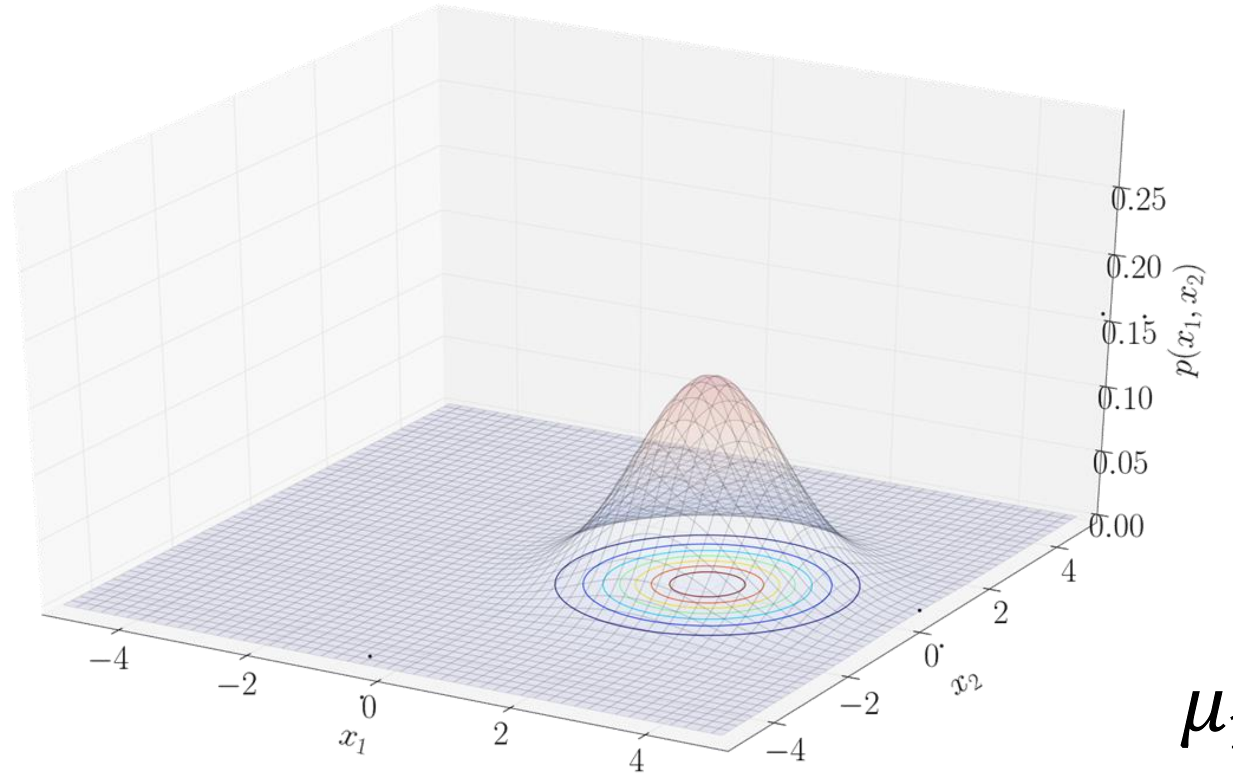
Mean

$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$

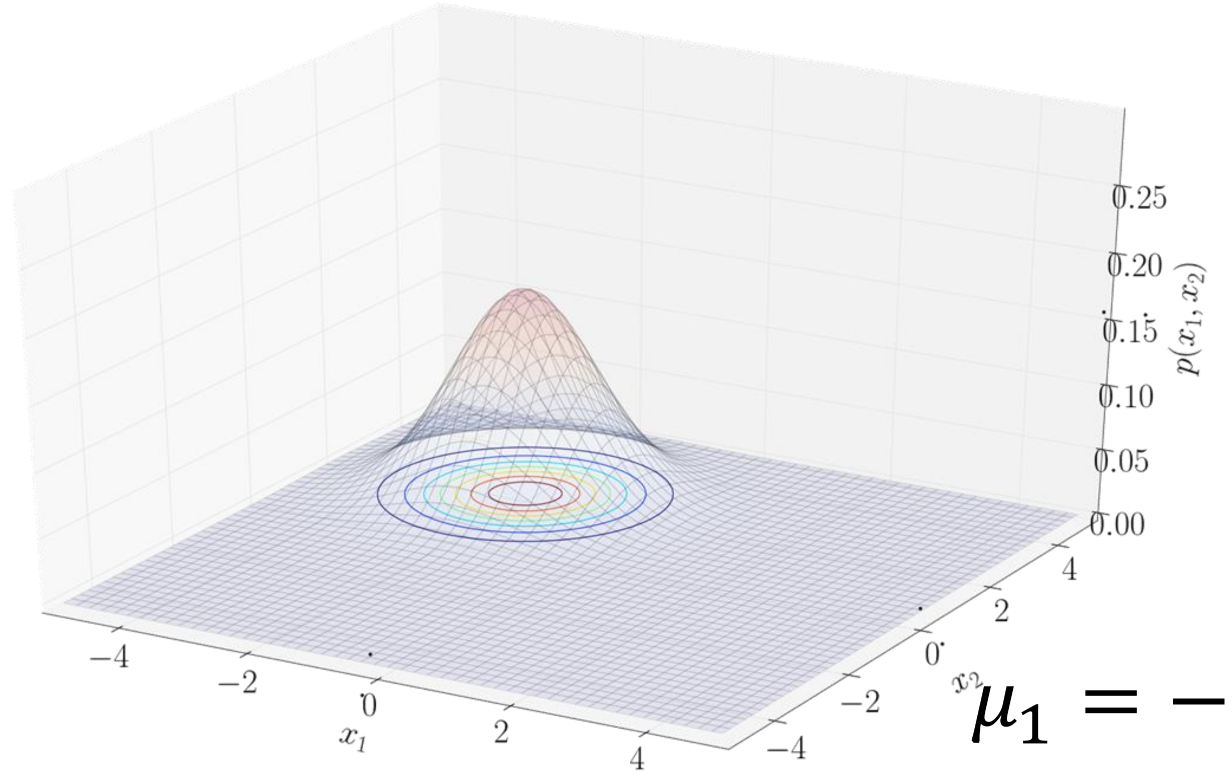
μ_1, μ_2 determine the location



$\mu_1 = 0, \mu_2 = 0$



$\mu_1 = 2, \mu_2 = 0$



$\mu_1 = -2, \mu_2 = 2$

Variance

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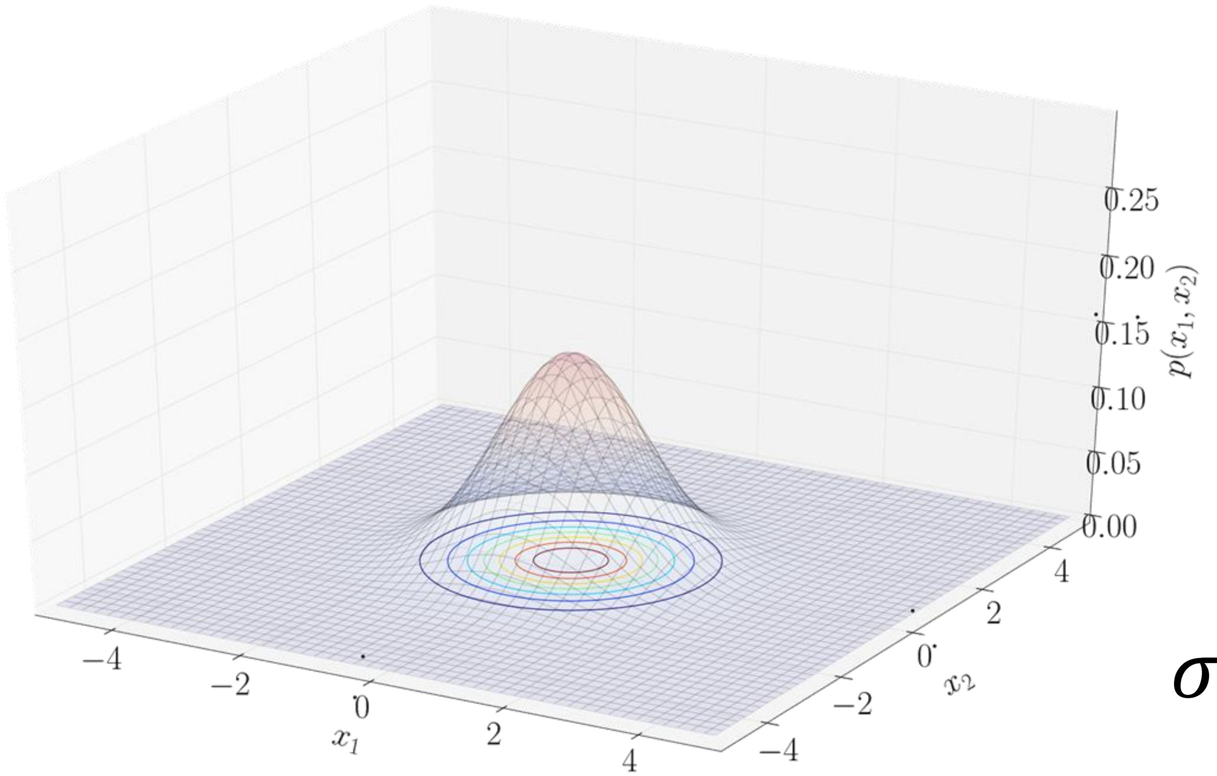
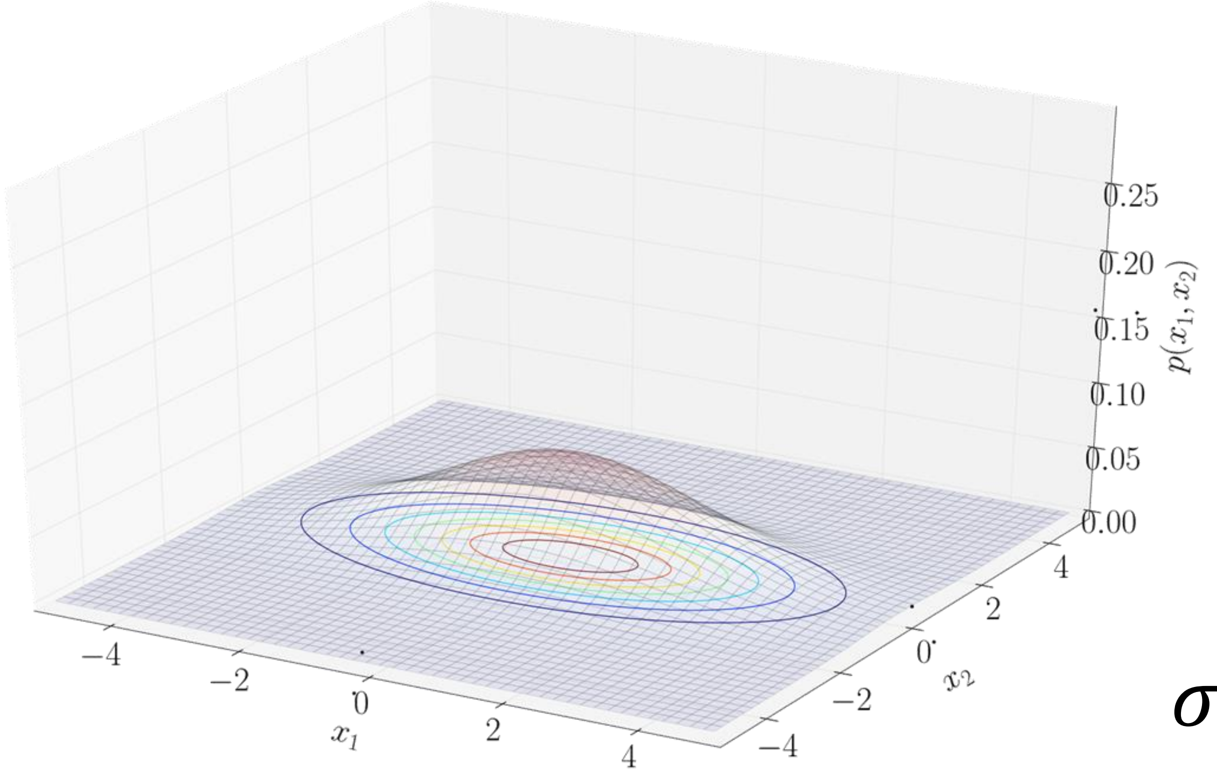
$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$

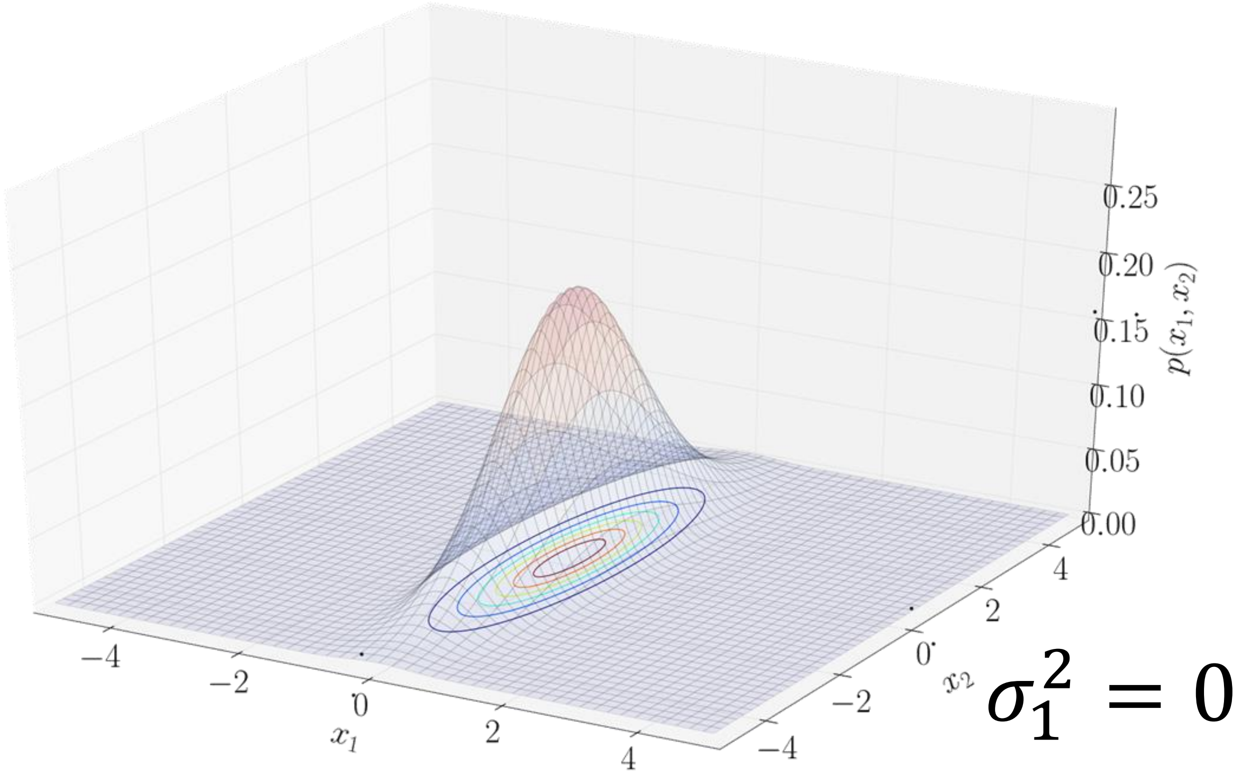
σ_1^2, σ_2^2: determine how spread out the values are in each direction.



$\sigma_1^2 = \sigma_2^2 = 1$



$\sigma_1^2 = 4, \sigma_2^2 = 1$

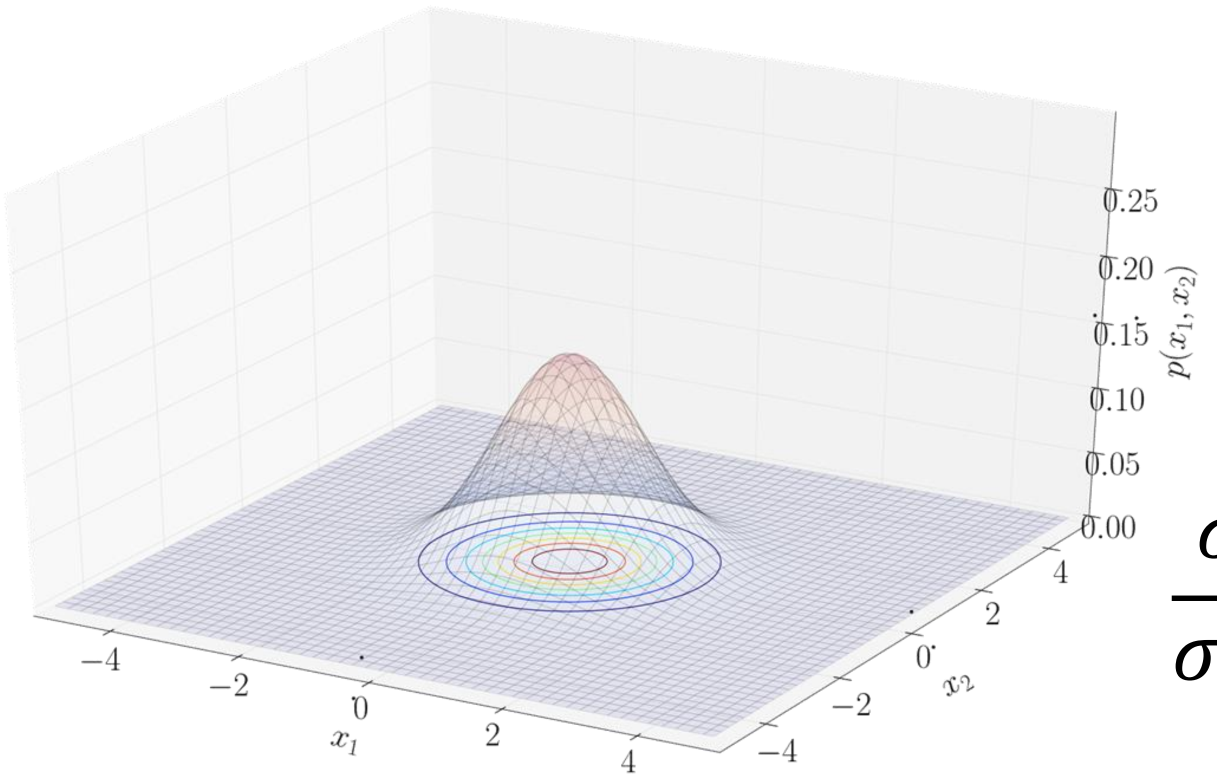


$\sigma_1^2 = 0.2 \; \sigma_2^2 = 3$

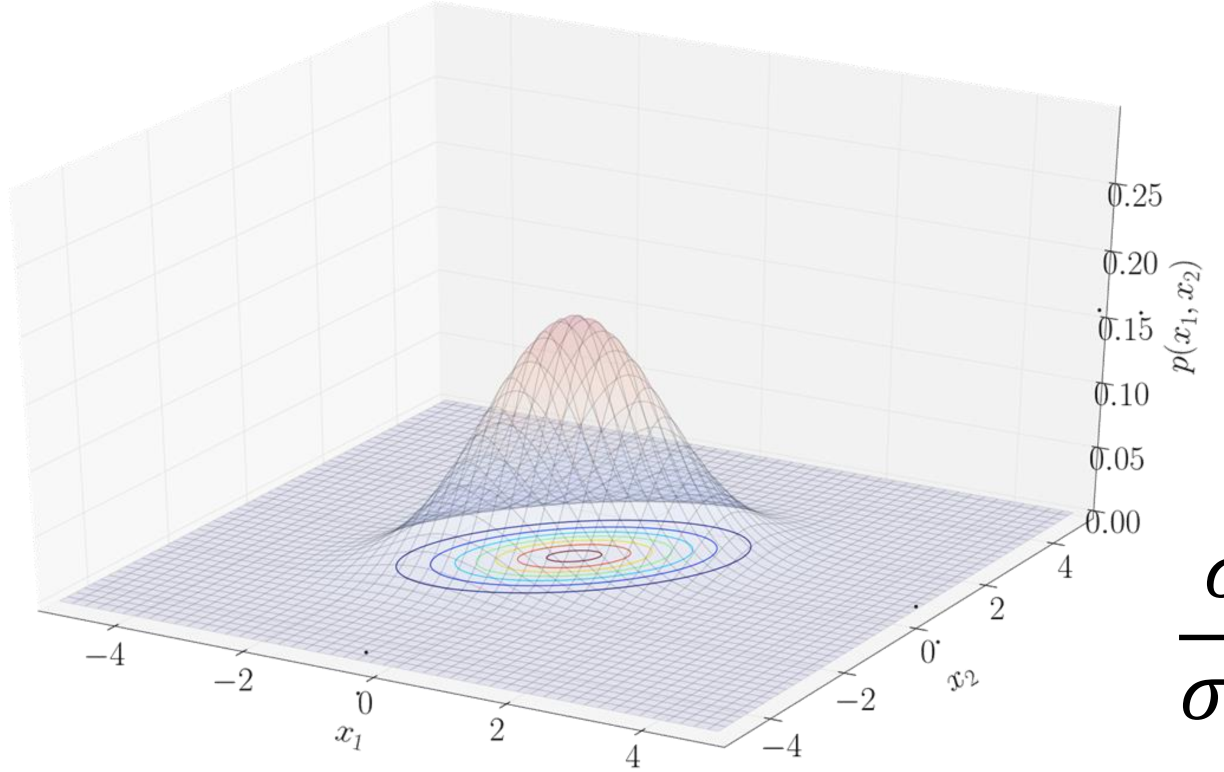
Covariance

$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$

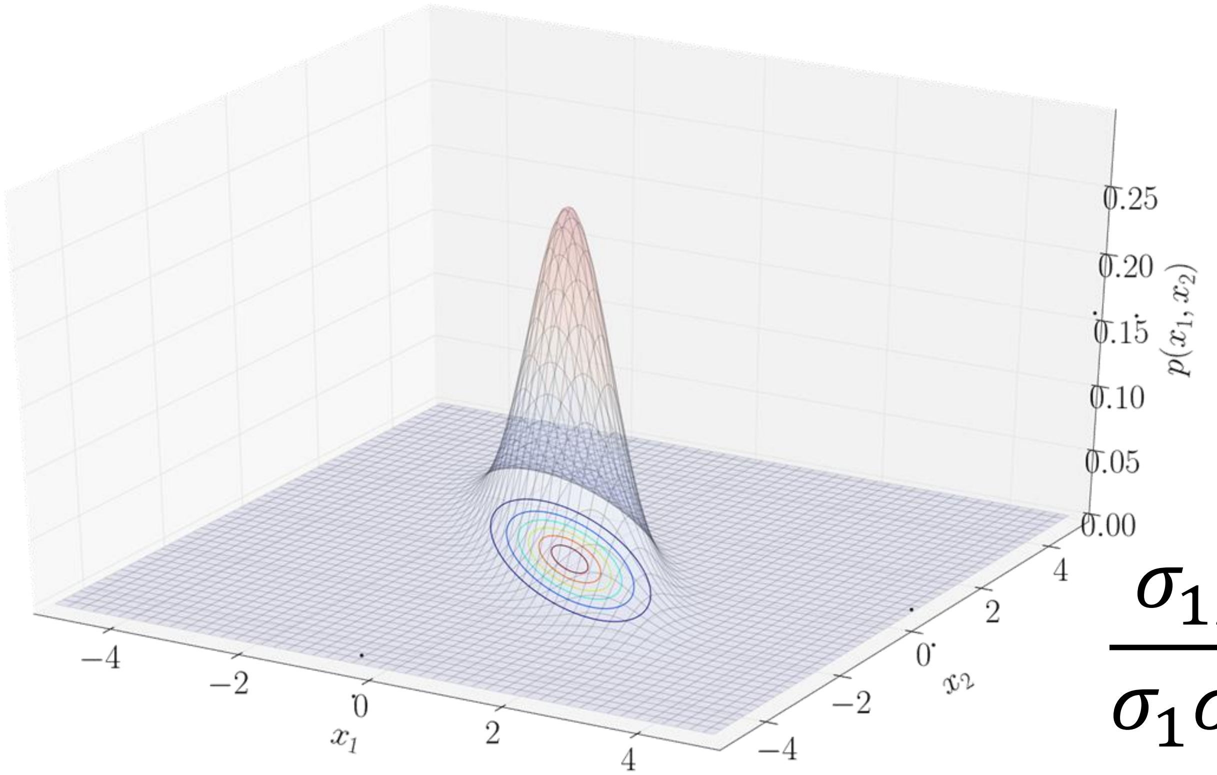
σ_{12} determines how much x_1 and x_2 change together.



$$\frac{\sigma_{12}}{\sigma_1 \sigma_2} = 0$$



$$\frac{\sigma_{12}}{\sigma_1 \sigma_2} = 0.5$$



$$\frac{\sigma_{12}}{\sigma_1 \sigma_2} = -0.9$$

Marginal and conditional distribution

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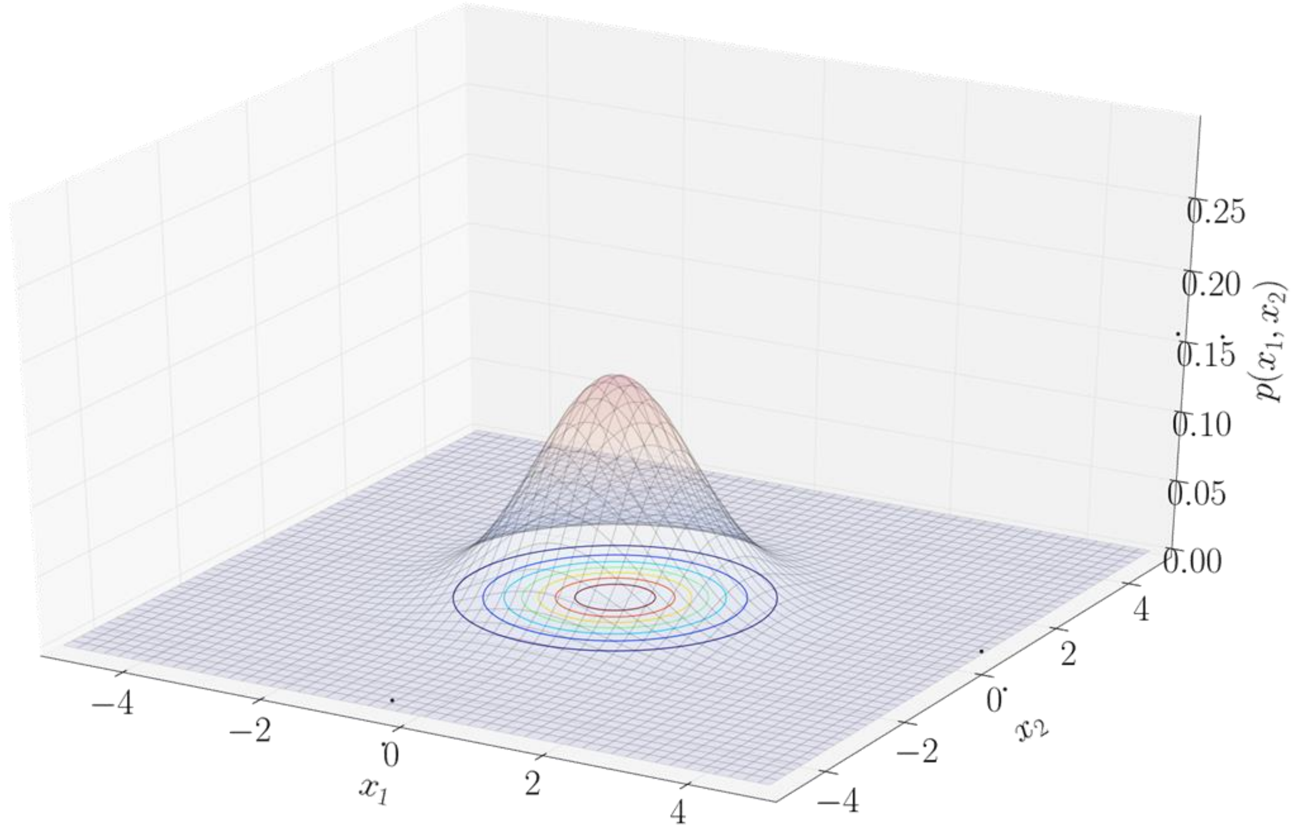
Bivariate normal:

$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$

What is the distribution of x_1…

- … if we don't know anything about x_2?
- … if we have observed the value of x_2?



Marginal and conditional distribution

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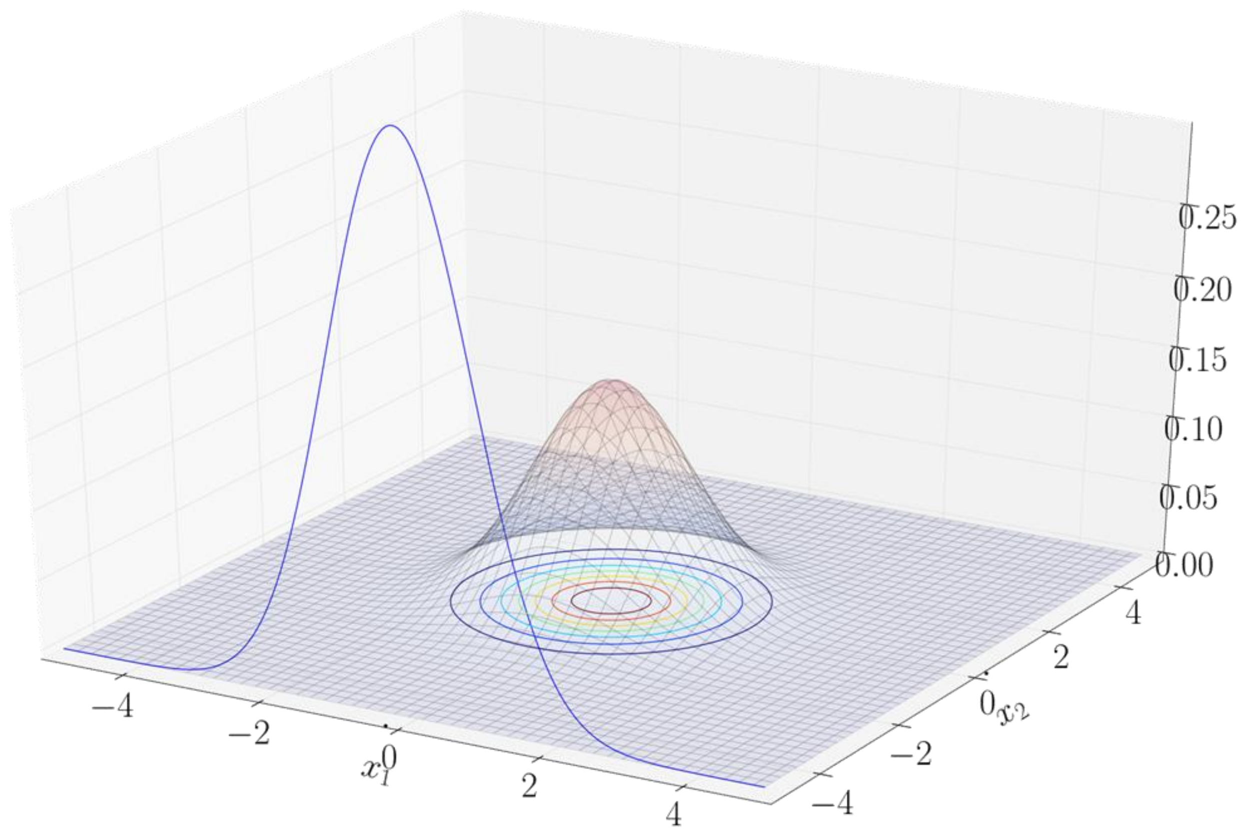
Bivariate normal:

$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$



What is the distribution of x_1…

- … if we don't know anything about x_2?
- … if we have observed the value of x_2?

Marginal distribution $p(x_1)$

Marginal and conditional distribution

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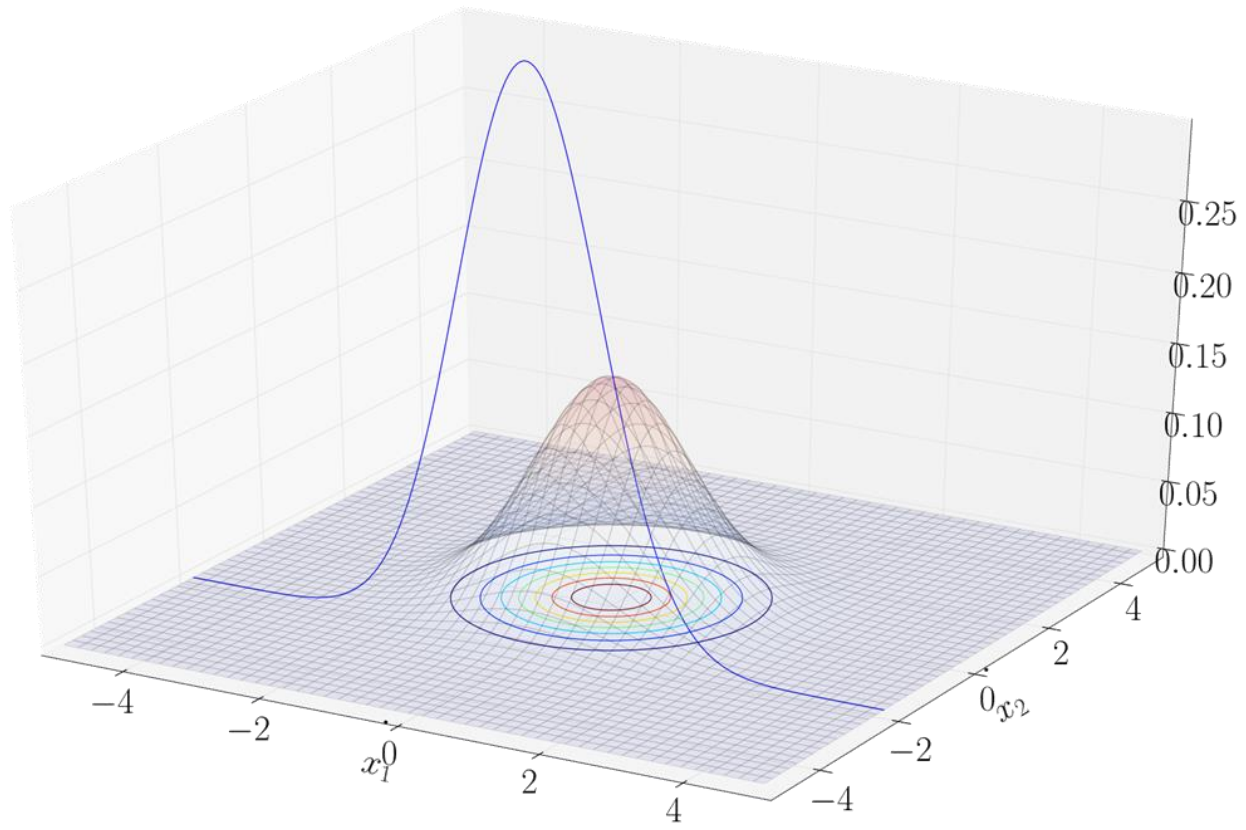
Bivariate normal:

$$p(x_1, x_2) = N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}\right)$$



What is the distribution of x_1…

- … if we don't know anything about x_2?
- … if we have observed the value of x_2?

Conditional distribution $p(x_1 | x_2 = \tilde{x}_2)$

Marginal and conditional distribution

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Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be jointly normal distributed random variables

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_m \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{x_1} \\ \vdots \\ \mu_{x_n} \\ \mu_{y_1} \\ \vdots \\ \mu_{y_m} \end{pmatrix}, \begin{pmatrix} \Sigma_{x_1 x_1} & \cdots & \Sigma_{x_1 x_n} & \Sigma_{x_1 y_1} & \cdots & \Sigma_{x_1 y_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Sigma_{x_n x_1} & \cdots & \Sigma_{x_n x_n} & \Sigma_{x_n y_1} & \cdots & \Sigma_{x_n y_m} \\ \Sigma_{y x_1} & \cdots & \Sigma_{y_1 x_n} & \Sigma_{y_1 y_1} & \cdots & \Sigma_{y_1 y_m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \Sigma_{y_m x_1} & \cdots & \Sigma_{y_m x_n} & \Sigma_{y_m y_1} & \cdots & \Sigma_{y_m y_m} \end{pmatrix} \right)$$

Marginal and conditional distribution

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Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be jointly normal distributed random variables

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix} \right)$$

Marginal and conditional distribution

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The marginal distribution is the normal distribution

$$p(x) = N(\mu_x, \Sigma_{xx}).$$

The conditional distribution is the normal distribution

$$p(x|y = \tilde{y}) = N(\overline{\mu}, \overline{\Sigma})$$

where

$$\overline{\mu} = \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(\tilde{y} - \mu_y)$$
$$\overline{\Sigma} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}$$

Summary

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$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{pmatrix}\right)$$

- Completely defined by mean and covariance matrix.
- Very flexible:
  - Defined by n + $n\frac{(n+1)}{2}$ parameters
  - Yet always unimodal and symmetric
- The marginal and conditional distributions are again normal distributions.