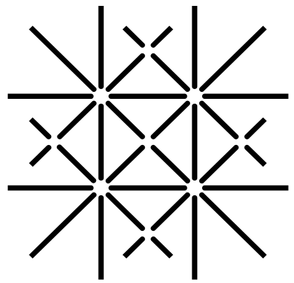


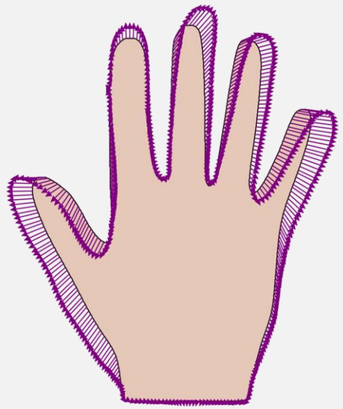
University
of Basel

Finite rank representation of a GP (advanced topic)

Finite observations revisited

University of Basel

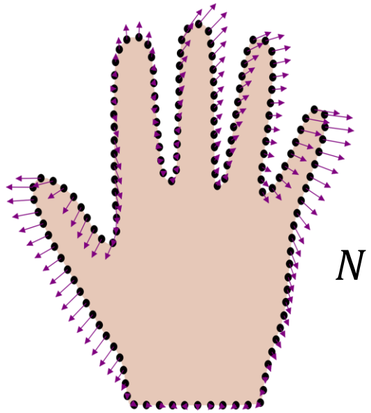
Continuous representation



$$GP(\mu, k)$$

Computer implementation

Discrete representation

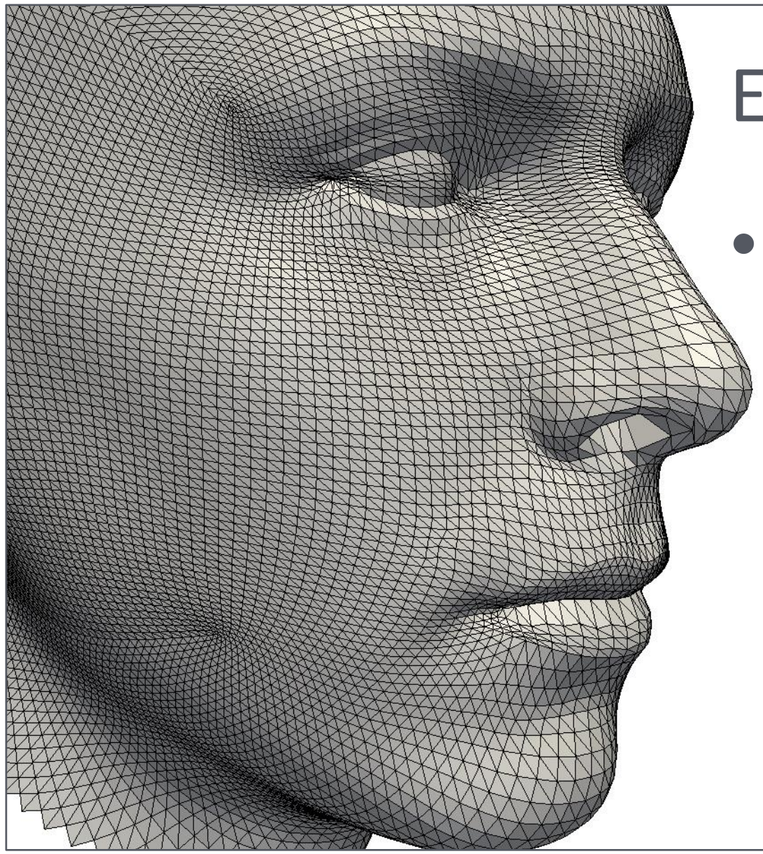


$$N\left(\begin{pmatrix}\mu_1(x_1)\\ \mu_2(x_1)\\ \vdots\\ \mu_1(x_n)\\ \mu_2(x_n)\end{pmatrix}, \begin{pmatrix}k_{11}(x_1,x_1) & k_{12}(x_1,x_1) & \cdots & k_{11}(x_1,x_n) & k_{12}(x_1,x_n)\\ k_{21}(x_1,x_1) & k_{22}(x_1,x_1) & & k_{21}(x_1,x_n) & k_{22}(x_1,x_n)\\ \vdots & & \ddots & & \vdots\\ k_{11}(x_n,x_1) & k_{12}(x_n,x_1) & & k_{11}(x_n,x_n) & k_{12}(x_n,x_n)\\ k_{21}(x_n,x_1) & k_{22}(x_n,x_1) & \cdots & k_{21}(x_n,x_n) & k_{22}(x_n,x_n)\end{pmatrix}\right)$$

A problem with GPs

University of Basel

- The resulting covariance matrix can be large



Example: Basel face model

- Mesh contains 53490 vertices

$$N\left(\begin{pmatrix}\mu(x_1)\\ \vdots \\ \mu(x_n)\end{pmatrix}, \underbrace{\begin{pmatrix}k(x_1,x_1) & \cdots & k(x_1,x_n)\\ \vdots & \ddots & \vdots \\ k(x_n,x_1) & \cdots & k(x_n,x_n)\end{pmatrix}}_{(53490\cdot 3)\times(53490\cdot 3)}\right)$$

The Karhunen-Loève expansion

University of Basel

We can write $\mathrm{u} \sim GP(\mu, k)$

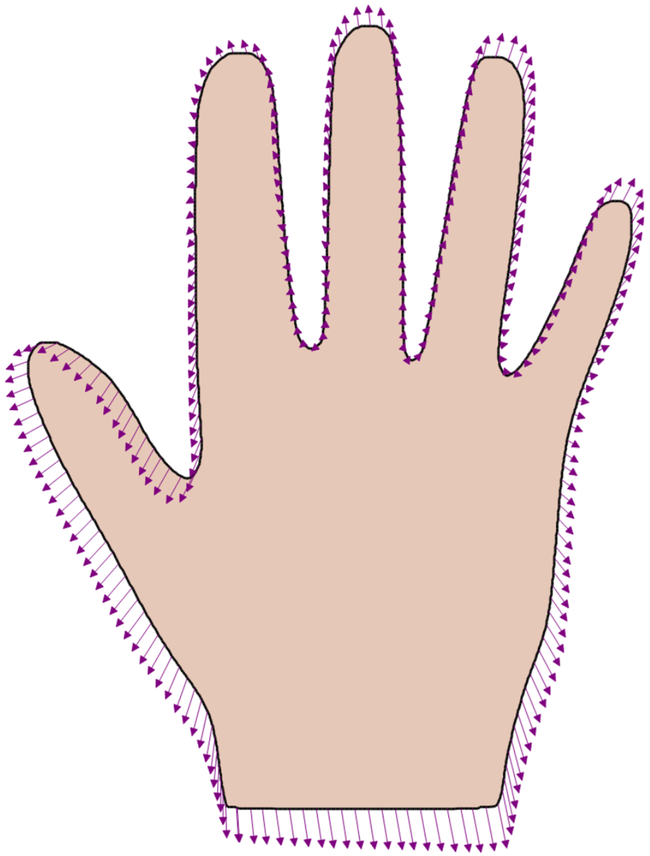
as $u \sim \mu + \sum_{i=1}^{\infty} \alpha_i \sqrt{\lambda_i}\, \phi_i,\ \alpha_i \sim N(0,1)$

- ϕ_i is the eigenfunction with associated eigenvalue λ_i of the linear operator

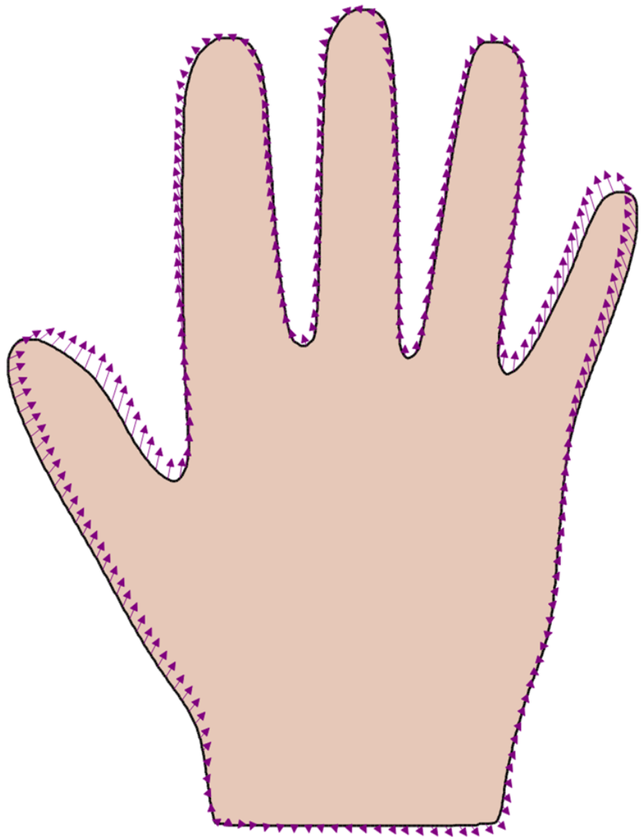
$$[T_k u](x) = \int k(x,s)u(s)ds$$

Eigenfunctions of the hand model

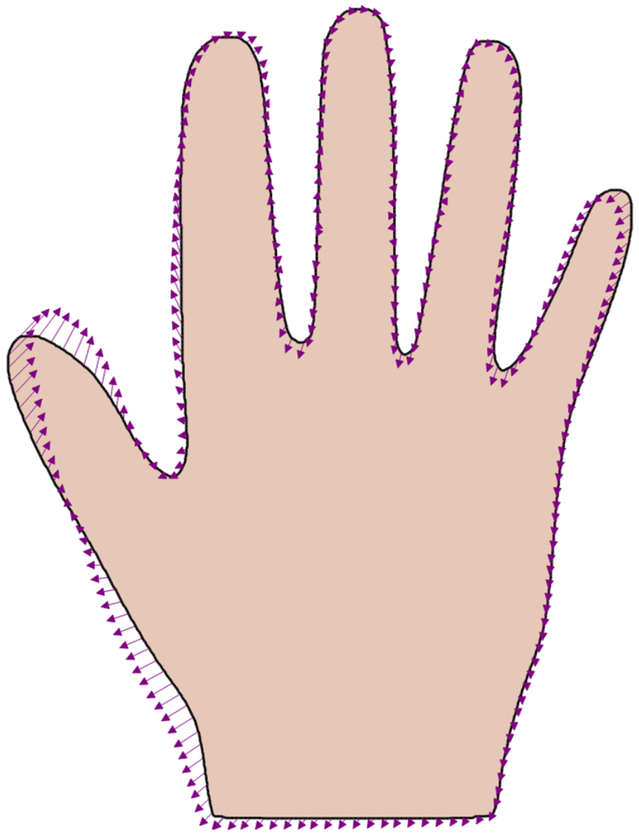
University of Basel



$\sqrt{\lambda_1}\phi_1$



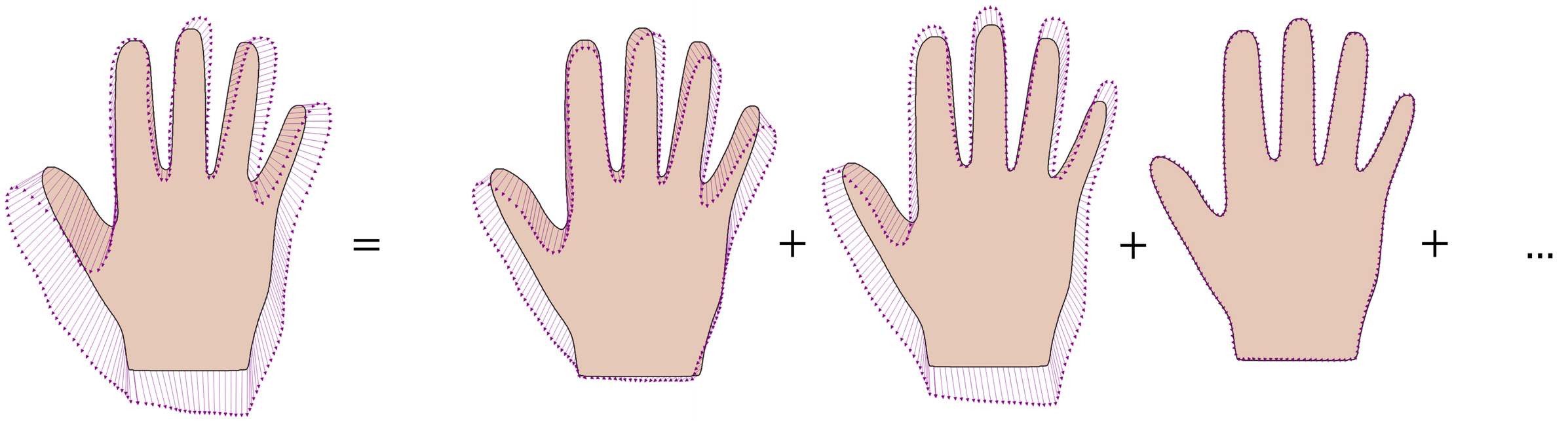
$\sqrt{\lambda_2}\phi_2$



$\sqrt{\lambda_3}\phi_3$

Karhunen-Loève expansion

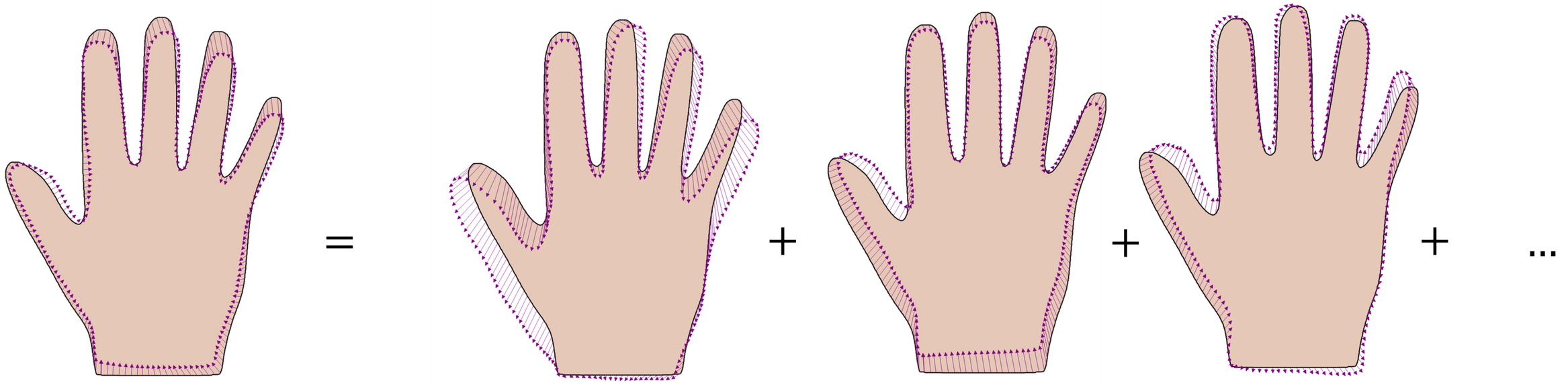
University of Basel



$\tilde{u}$ μ $2.1\sqrt{\lambda_1}\phi_1$ $0.3\sqrt{\lambda_2}\phi_2$

Karhunen-Loève expansion

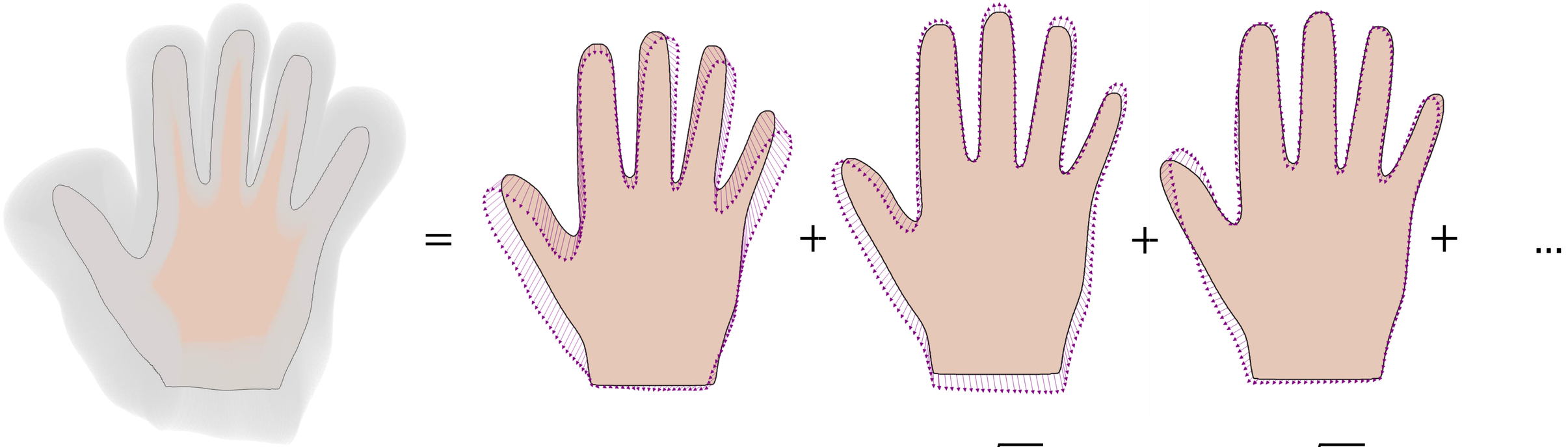
University of Basel



$\tilde{u}$ μ $-1.2\sqrt{\lambda_1}\phi_1$ $1.8\sqrt{\lambda_2}\phi_2$

Karhunen-Loève expansion

University of Basel



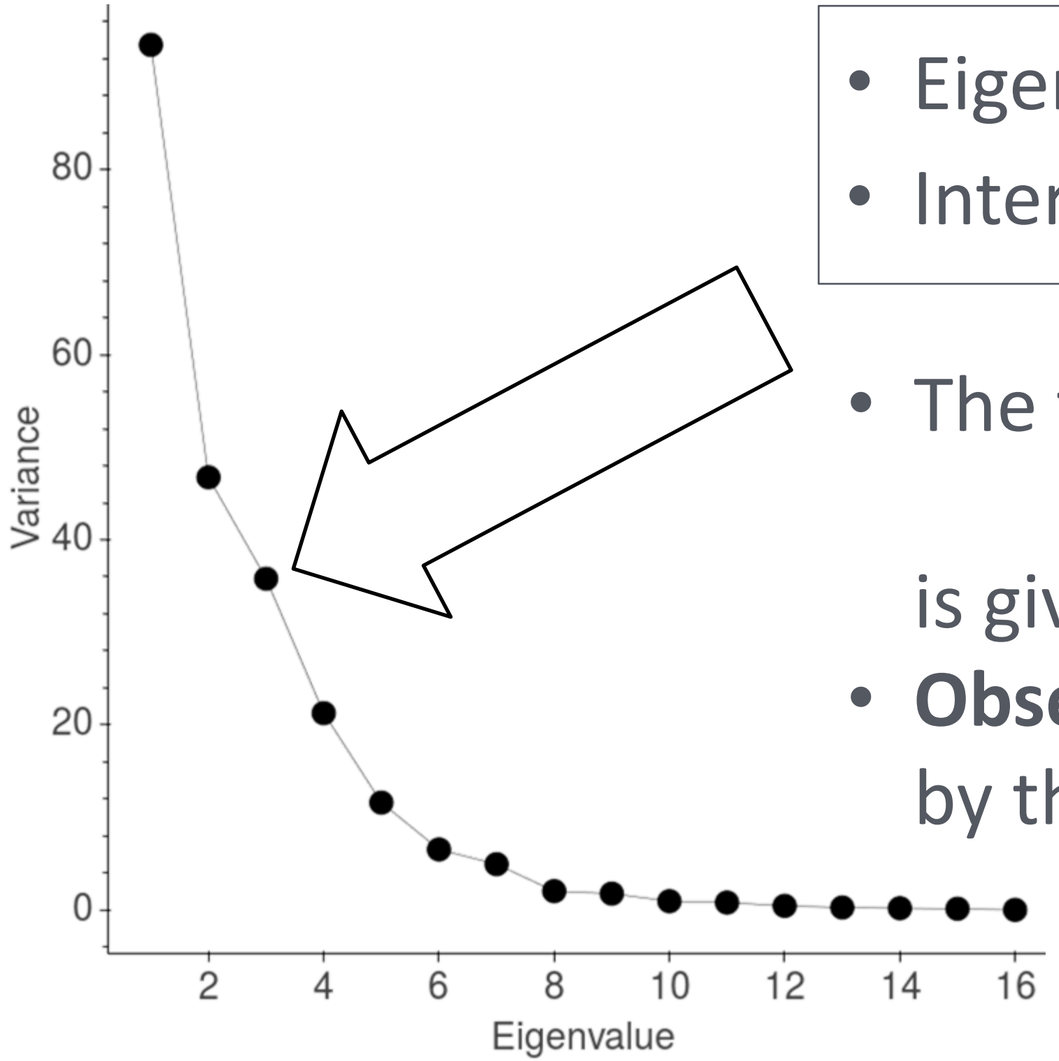
$\tilde{u} \sim GP(\mu, k)$ = μ + $\alpha_1 \sqrt{\lambda_1}\phi_1$ + $\alpha_2 \sqrt{\lambda_2}\phi_2$ + ...

$\alpha_1 \sim N(0,1)$

$\alpha_2 \sim N(0,1)$

Eigenvalues and variance

University of Basel



- Eigenvalue λ_i
- Interpretation: Variance of $\alpha_i \sqrt{\lambda_i} \phi_i$

- The total variance of the process
$$u \sim GP(\mu, k)$$
is given by $\sum_{i=1}^{\infty} \lambda_i$.
- **Observation**: Most variance is explained by the first eigenfunctions

Low-rank approximation

University of Basel

$$u = \mu + \sum_{i=1}^{r} \alpha_i \sqrt{\lambda_i}\, \phi_i, \qquad \alpha_i \sim N(0,1)$$

Main idea: Represent process using only the first r components

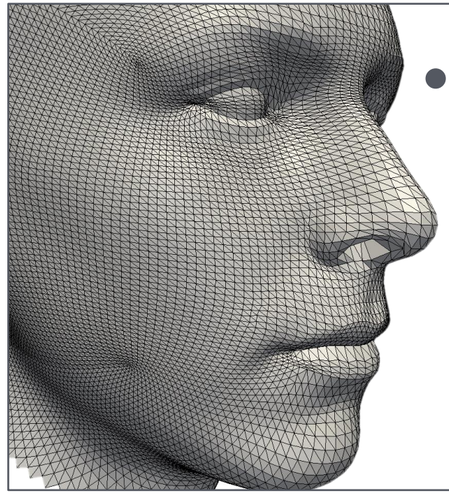
- We have a finite, parametric representation of the process.
- Any deformation u is determined by the coefficients $\alpha = (\alpha_1, \ldots, \alpha_r)$

$$p(u) = p(\alpha) = \prod_{i=1}^{r} \frac{1}{\sqrt{2\pi}} \exp(-\alpha_i^2/2)$$

Computational aspects

University of Basel

- Numerical methods are used for computing (λ_i, ϕ_i)
  - Very efficient for models that are learned from examples
  - Possible for general models
  - The full covariance matrix is never computed



- If rank r is low, the representation is efficient
  - can represent shapes with millions of points